Math 821, Spring 2013, Lecture 13

Karen Yeats
(Scribe: Mahdieh Malekian)

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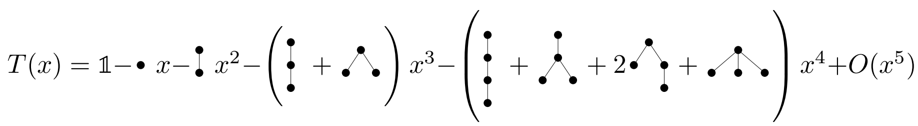
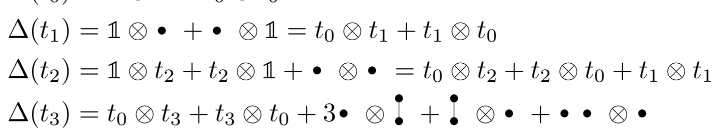
1 SubHopf Algebra Results

Example. Last time we showed that $T(x) = \mathbb{1} - B_+(\frac{1}{T(x)})$.

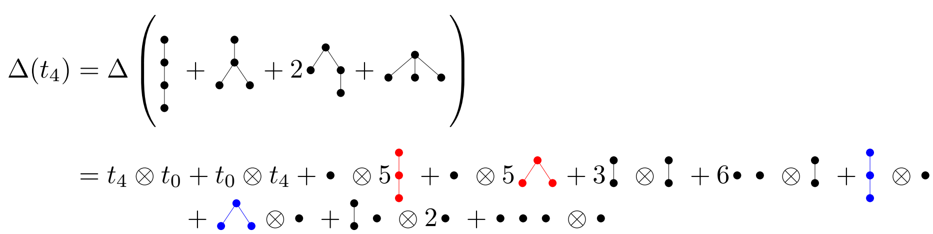


Define $t_0 = \mathbb{1}, n > 0, t_n = -[x^n]T(x)$. Now what is $\Delta(t_i)$?

$\Delta(t_0) = \mathbb{1} \otimes \mathbb{1} = t_0 \otimes t_0$



$= t_0 \otimes t_3 + t_3 \otimes t_0 + 3t_1 \otimes t_2 + (t_2 + t_1^2) \otimes t_1$



$= t_4 \otimes t_0 + t_0 \otimes t_4 + 5t_1 \otimes t_3 + 3(t_2 + 2t_1^2) \otimes t_2 + (t_3 + 2t_2t_1 + t_1^3) \otimes t_1$

What we are observing is if A be the algebra generated by the t_i then $\Delta(t_i) \subseteq A \otimes A$. So A is not just a subalgebra of the Connes-Kreimer Hopf algebra; it is also a subHopf algebra.

Theorem 1 *[1] Let H be a graded connected Hopf algebra which is either free* *(words over generators) or free-commutative (polynomial algebra over generators) as an algebra. Let $(B_+^n)_{n=1}^\infty$ be a family of Hochschild 1-cocycles. Then*

$$T(x) = \mathbb{1} + \sum_{n=1}^{\infty} x^n w_n B_+^n(T(x)^{n+1}),$$

where the w_n are from k. This has a unique solution given recursively and $\Delta t_n = \sum_{j=0}^{n} P_{n,j}(t_0, \ldots, t_{n-j}) \otimes t_j$, where $T(x) = \sum_{n=0}^{\infty} t_n x^n$ and $P_{n,j}$ is a polynomial.

But this isn't quite satisfactory because the specification is rather special. If we only have a B_+ of weight 1, here's a nice theorem:

Theorem 2 *[2] Let $P = \sum_{n=0}^{\infty} p_n x^n$ be a formal power series with $p_0 = 1$, then $T(x) = xB_+(P(T(x)))$ has a unique solution given recursively and the following are equivalent:*

1. *The algebra generated by the t_i ($T(x) = \sum_{n=0}^{\infty} t_n x^n$) is a subHopf algebra.*
2. *$\exists (\alpha, \beta) \in \mathbb{Q}^2$ such that $(1 - \alpha\beta x)P'(x) = \alpha P(x)$.*
3. *$\exists (\alpha, \beta) \in \mathbb{Q}^2$ such that*
   - *(a) $P(x) = 1$ if $\alpha = 0$.*
   - *(b) $P(x) = e^{\alpha x}$ if $\beta = 0, a \neq 0$.*
   - *(c) $P(x) = (1 - \alpha\beta x)^{-\frac{1}{\beta}}$ else.*

Together

Theorem 3 *[3] Suppose*

$$T(x) = \sum_{j \in J} x^j \beta^j + (P_j(T(x))) \qquad (*)$$

with $J \subseteq \{1, 2, \ldots\}, P_j(0) = 1, P_j$ formal power series, and suppose the coefficients of the solution $T(x)$ form a subHopf algebra. Then one of the following holds:

1. *$\exists \lambda, \mu \in \mathbb{Q}$ such that $(*)$ is*

$$T(x) = \sum_{j \in J} x^j B_+^j((1 - \mu T(x) Q(T(x))^j),$$

where

$$Q(h) = \begin{cases} (1 - \mu\lambda)^{\frac{\lambda}{\mu}} & \text{if } \mu \neq 0 \\ e^{\lambda h} & \text{if } \mu = 0. \end{cases}$$

2. $\exists m \geq 0$ *and* $\alpha \in \mathbb{Q}, \alpha \neq 0$ *such that* $(*)$ *is*

$$T(x) = \sum_{\substack{j \in J \\ m \mid j}} x B_+^j (1 + \alpha T(x)) + \sum_{\substack{j \in J \\ m \nmid j}} x B_+^j (\mathbb{1})$$

Similar results hold for specification which are systems.

Let's prove part of Foissy's 2007 result. We'll do (1) $\Rightarrow$ (2) $\Rightarrow$ (3). The proof of (3) $\Rightarrow$ (1) goes through the plane version and involves looking at reductions on the pairs (α, β).

Proof.

(2) $\Rightarrow$ (3) Solve the differential equation. If $\alpha = 0$ the differential equation becomes $P'(x) = 0$, so P is a constant and so by the normalization $P(x) = 1$.

Assume $\alpha \neq 0$. If $\beta = 0$, $P'(x) = \alpha P(x)$ so $P(x) = e^{\alpha x}$ (and normalize this way as $P(0) = 1$).

Assume α, β are both nonzero. Then the differential equation is $(1 - \alpha\beta x)\frac{dP}{dx} = \alpha P(x)$, so

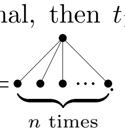
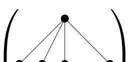
$$\frac{dP}{P(x)} = \frac{\alpha dx}{1 - \alpha\beta x},$$

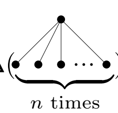
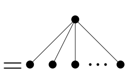
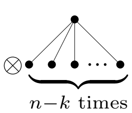
so

$$\log P + C = \int \frac{dP}{P} = \int \frac{\alpha dx}{1 - \alpha\beta x} = \log(1 - \alpha\beta x).$$

But $P(0) = 1$, so $0 + C = 0$, so $C = 0$. Therefore $\log P = \log(1 - \alpha\beta x)^{-\frac{1}{\beta}}$, so $P(x) = (1 - \alpha\beta x)^{-\frac{1}{\beta}}$.

(1) $\Rightarrow$ (2) Let A be the algebra generated by the coefficients of T. First note that if $P(x) = 1$ then $T(x) = x$ and all is true. So from now on assume $P(x)$ has a constant term.

Suppose $p_n \neq 0, n \geq 2, n$ minimal, then $t_1 = t_2 = \ldots = t_n = 0$ but $t_{n+1} = p_n B_+(\bullet^{\,n})$, but $B_+(\bullet^{\,n}) =$ . But by hypothesis we have a subHopf algebra, so Δ () $\subseteq A \otimes A$. But A contains no trees of size $2 - n$, which is a contradiction since

$\Delta($  $) =$  $\otimes \mathbb{1} + \sum_{k=0}^{n} \binom{n}{k} \bullet^{\,k} \otimes$ .

So we need to have $t_{n-k} \neq 0$, $\forall 0 \leq k \leq n$, so $p_1 \neq 0$. As a further consequence there is a tree of every size in A, because $0 \neq p_1 B_+(t_n) \in$

H_{n+1} appears in t_{n+1}, where H is the Connes-Kreimer Hopf algebra $/\mathbb{Q}$. Let $Z : H \longrightarrow \mathbb{Q}$ be the characteristic map of $\bullet$, i.e. $Z(F) = \delta_{\bullet, F}$ on forests and extended linearly. Consider $(Z \otimes id)\Delta(T(x))$. By assumption A is a subHopf algebra so $\Delta(t_n) \subseteq A \otimes A$, so $(Z \otimes id)\Delta(T(x)) \in A[[x]]$. Also observe the following

$$(Z \otimes id)(ab) = (Z \otimes id)(a)(\varepsilon \otimes id)(b) + (\varepsilon \otimes id)(a)(Z \otimes id)(b), \qquad (1)$$

for $a, b \in H \otimes H$. Let's check this. It suffices to check for a, b pure tensors, i.e. $a = a_1 \otimes a_2, b = b_1 \otimes b_2$.

$$\begin{aligned} \text{LHS of (1)} &= (Z \otimes id)(a_1 b_1 \otimes a_2 b_2) \\ &= \begin{cases} a_2 b_2 & \text{if } a_1 = \bullet, b_1 = \mathbb{1} \text{ or } a_1 = \mathbb{1}, b_1 = \bullet \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

On the RHS of (1), again if $a_1 \neq \bullet, b_1 \neq \bullet$, we get 0. If $a_1 = \bullet$ then we have $a_2 \varepsilon(b_1) b_2 + 0$, since $\varepsilon(a_1) = 0$, so for this to be nonzero we need $b_1 = \mathbb{1}$. Similarly for the second term on the RHS. (Note throughout I have pushed scalars on to the a_2, b_2 part.) This proves (1).

$$\begin{aligned} (Z \otimes id)\Delta(T(x)) &= (Z \otimes id)\Delta(xB_+(P(T(x))) \\ &= (Z \otimes id)\Delta(\sum_{n=0}^{\infty} xp_n B_+(T(x)^n)) \\ &= \sum_{n=0}^{\infty} (Z \otimes id)\Delta B_+(T(x)^n) \\ &= \sum_{n=0}^{\infty} xp_n Z(B_+(T(x)^n)) + \sum_{n=1}^{\infty} xp_n (Z \otimes B_+)\Delta(T(x)^n)). \end{aligned}$$

(Since $\Delta B_+ = B_+ \otimes \mathbb{1} + (id \otimes B_+)\Delta$, we have:)

$$= Z(T(x)) + B_+(\sum_{n=1}^{\infty} xp_n (Z \otimes id)\Delta(T(x))^n),$$

since the constant does not affect B_+.

$$\begin{aligned} &= t_1 + B_+ \left(\sum_{n=0}^{\infty} xnp_n (\varepsilon \otimes id)\Delta(T(x))^{n-1} (Z \otimes id)\Delta(T(x)) \right) \text{ by (1)} \\ &= t_1 + B_+ \left(\sum_{n=0}^{\infty} xnp_n T(x)^{n-1} (Z \otimes id)\Delta(T(x) \right) \\ &= t_1 + xB_+(P^{'} T(x))(Z \otimes id)\Delta(T(x))). \end{aligned}$$

Next let

$$\begin{aligned} L : H[[x]] &\longrightarrow H[[x]] \\ a &\longmapsto xB_+(P^{'}(T(x))a). \end{aligned}$$

L increases degree, so $id - L$ is formally invertible. So the calculation above says

$$(id - L)((Z \otimes id)\Delta(T(x))) = Z(T(x))(t_1),$$

or, equivalenty

$$(Z \otimes id)\Delta(T(x)) = (id - L)^{-1}(t_1) = t_1(id - L)^{-1}(\mathbb{1}).$$

Now since A is a subHopf algebra, we have

$$\begin{aligned}(Z \otimes id)\Delta(T(x)) &\subseteq A[[x]] \\ \Rightarrow (id - L)^{-1}(\mathbb{1}) &\subseteq A[[x]].\end{aligned}$$

Now the third step is to pull out easy coefficient and compare. From $T(x) = xB_+(P(T(x)))$ we have the recursive expression

$$t_1 = \bullet\ , t_{n+1} = \sum_{k=1}^{n} \sum_{\alpha_1+\ldots+\alpha_k=n} p_k B_+(t_{\alpha_1} \ldots t_{\alpha_k}).$$

Write $(id - L)^{-1}(\mathbb{1}) = \sum_{i=0}^{\infty} b_i x^i$. By induction

$$\begin{aligned} b_0 &= 1 \\ b_{n+1} &= \sum_{k=1}^{n} \sum_{\alpha_1+\ldots+\alpha_k=n} (k+1)p_{k+1} B_+(t_{\alpha_1} \ldots t_{\alpha_k}) \\ &\quad + \sum_{k=1}^{n} \sum_{\alpha_1+\ldots+\alpha_k=n} k p_k B_+(b_{\alpha_1} t_{\alpha_2} \ldots t_{\alpha_k}). \end{aligned}$$

Now compare coefficients. Consider $B_+(f_n), B_+(b_n)$, i.e.trees where the root has only one child and degree $n+1$ in f_{n+1} and b_{n+1}. Coefficient in f_{n+1} is $p_1 B_+(f_n)$, and the coefficient in b_{n+1} is $2p_2 B_+(f_n) + p_1 B_+(b_n)$, but by assumption the f_n make a subHopf algebra, and the b_n are in it, so $b_{n+1} = \lambda_{n+1} f_{n+1}, \lambda_{n+1} \in \mathbb{Q}$. So

$$\lambda_1 = p_1, \lambda_{n+1} = \left(\frac{2p_2}{p_1} + \lambda_n \right). \tag{2}$$

Consider $B_y(\bullet\ ^n)$ in f_{n+1} and b_{n+1}. In f_{n+1} we get p_n, and in b_{n+1} we get $(n+1)p_{n+1} + np_n p_1$, so

$$\lambda_{n+1} p_n = (n+1)p_{n+1} + np_n p_1, \quad \forall n \geq 1,$$

which together with (2) gives

$$(n+1)p_{n+1} + (p_1 - 2\frac{p_2}{p_1})np_n = p_1 p_n.$$

If we rewrite this at level of series we get

$$P^{'}(h) + (p_1 - 2\frac{p_2}{p_1})hP^{'}(h) = p_1 P(h).$$

Now let $\alpha = p_1, \beta = 2\frac{p_2}{p_1^2} - 1$ to get the result.

$\mathbb{A}$

References

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[2] L. Foissy. Fa di bruno subalgebras of the hopf algebra of planar trees from combinatorial dyson-schwinger equations. *Adv. Math*, pages 136–162, 2007.

[3] L. Foissy. General dyson-schwinger equations and systems. 2011.